Tree-Like Justification Systems are Consistent

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International Conference on Logic Programming
August 3rd, 2022

ARTIFICIAL
INTELLIGENCE
RESEARCH GROUP

OUR RESULT

Theorem

Every tree-like justification system is consistent.

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This result will be important for the next talk (“On Nested Justification Systems”).

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JUSTIFICATION THEORY: A UNIFYING FRAMEWORK

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- Logic Programming
- Abstract Argumentation
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The only thing to define is an evaluation of branches

Core idea: semantics is defined in terms of explanations why facts hold.
Not just relevant for theory; justifications show up in unexpected places

DEFINITIONS: JUSTIFICATION FRAMES

Definition

A justification frame $\mathbb{JF}$ is a tuple $\langle \mathbb{F}, \mathbb{F}_d, R \rangle$ with:

fact space $\mathbb{F}$ with $\mathcal{L} = \{\mathbf{t}, \mathbf{f}, \mathbf{u}\} \subseteq \mathbb{F}$.

Involution $\sim$ on $\mathbb{F}$
(with $\sim\mathbf{t} = \mathbf{f}$, $\sim\mathbf{f} = \mathbf{t}$, $\sim\mathbf{u} = \mathbf{u}$)

defined facts $\mathbb{F}_d \subseteq \mathbb{F} \setminus \mathcal{L}$; $\sim\mathbb{F}_d = \mathbb{F}_d$.

rules $R \subseteq \mathbb{F}_d \times 2^{\mathbb{F}}$

Example

$$\mathbb{F} = \{p, \sim p, q, \sim q, r, \sim r, s, \sim s, \mathbf{t}, \mathbf{f}, \mathbf{u}\}$$

$$\mathbb{F}_d = \{p, \sim p, q, \sim q, r, \sim r\}$$

$$p \leftarrow \sim q$$
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	Complementation:
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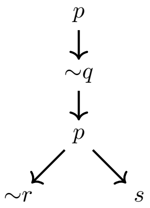
Let $\mathbb{JF} = \langle \mathbb{F}, \mathbb{F}_d, R \rangle$ be a justification frame. A (tree-like) justification J in $\mathbb{JF}$ is a labeled tree such that the set of children of each node is a case (rule) in R for that node.



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J is locally complete if all leaves are open

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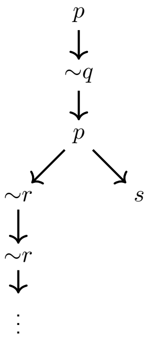

DEFINITIONS: BRANCH EVALUATION

Definition

A branch evaluation $\mathcal{B}$ maps every branch (sequence of facts) to an element of $\mathbb{F}$.

A justification system $\mathbb{JS}$ is a tuple $\langle \mathbb{F}, \mathbb{F}_d, R, \mathcal{B} \rangle$.

The stable branch evaluation $\mathcal{B}_{\mathrm{st}}$ maps

- finite branches to their last element
- infinite branches with positive tail to $\mathbf{f}$
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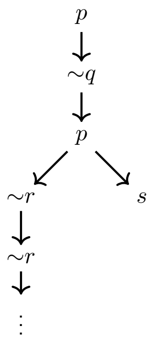

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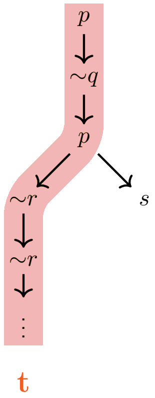

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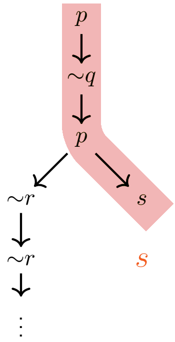

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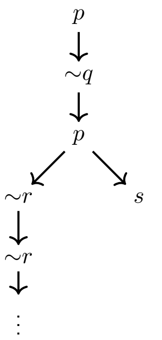

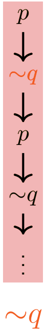

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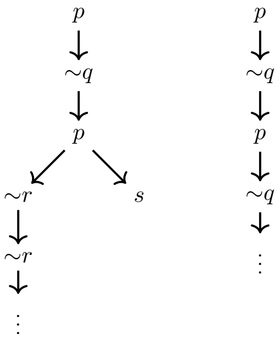

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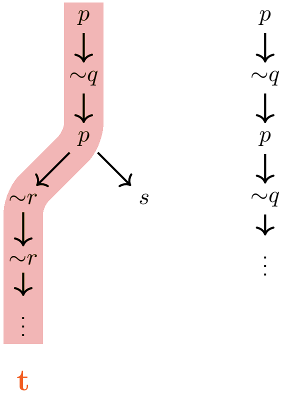

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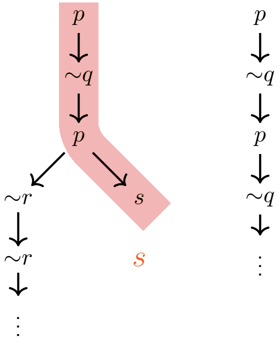

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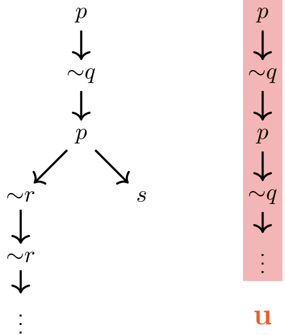

DEFINITIONS: INTERPRETATIONS

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An interpretation I maps each fact in $\mathbb{F}$ to a truth value (in $\mathcal{L}$) and

- Commutes with negation: $I(\sim x) = \sim I(x)$
- Identity on $\mathcal{L}$

$$I(p) = \mathbf{t}, I(q) = \mathbf{f}, I(r) = \mathbf{t}, I(s) = \mathbf{f}$$

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And implicitly: $I({\sim}p) = \mathbf{f}$, $I({\sim}q) = \mathbf{t}$,
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DEFINITIONS: SUPPORTED VALUE

order: $\mathbf{f} \leq_t \mathbf{u} \leq_t \mathbf{t}$

Definition

The value of node x in J under I is the value of the worst branch starting in x:

$$\mathrm{val}_{\mathcal{B}}(J, x, I) = \min_{\mathbf{b} \in B_J(x)} I(\mathcal{B}(\mathbf{b}))$$

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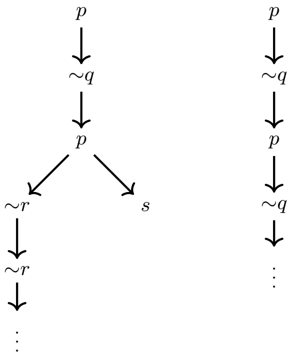

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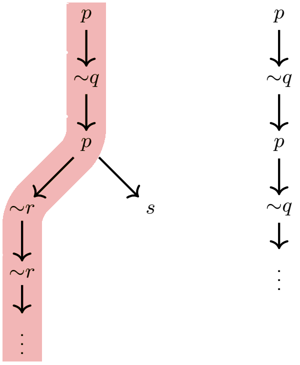

$I(\mathbf{t}) = \mathbf{t}$

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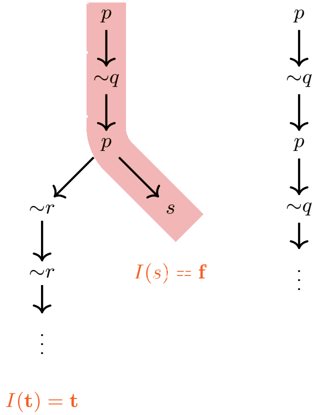

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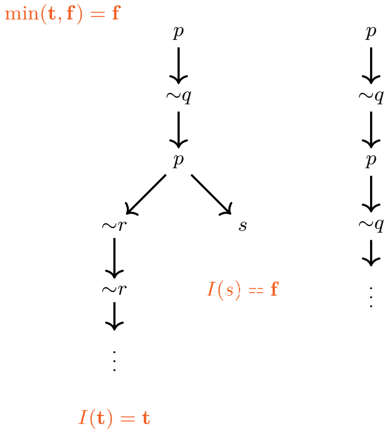

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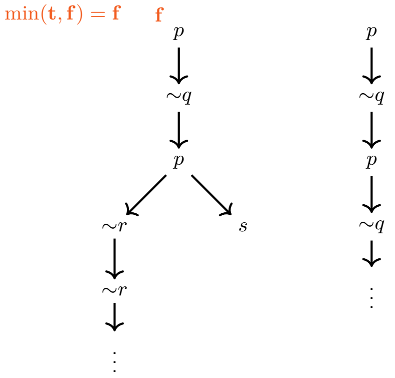

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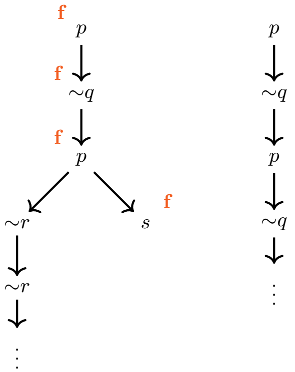

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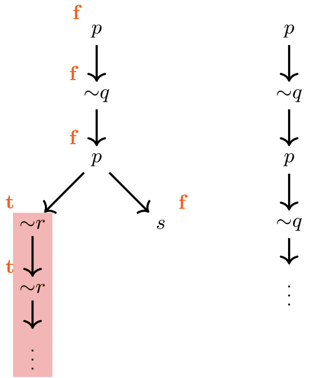

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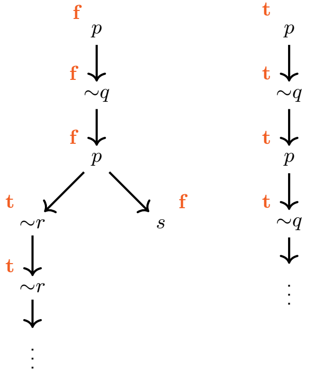

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The supported value of x under I is the value of the best justification for x:

$$\mathrm{SV}_{\mathcal{B}}(x, I) = \max_{J \in \mathfrak{J}(x)} \mathrm{val}(J, x, I)$$

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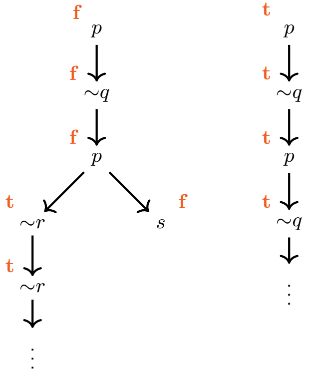

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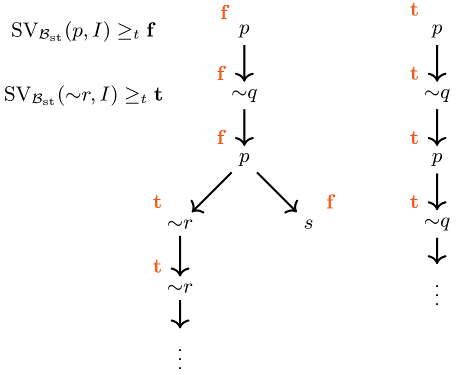

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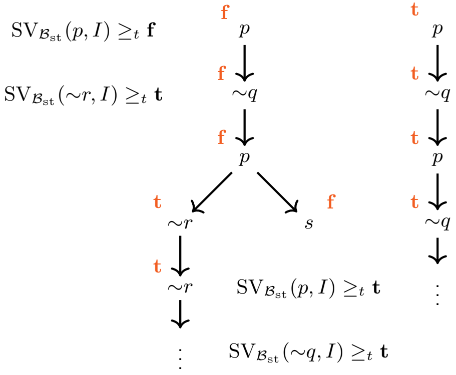

DEFINITIONS: MODEL

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An interpretation I is a $\mathcal{B}$-model if

$$I(x) = \mathrm{SV}_{\mathcal{B}}(x, I)$$

for all defined facts x.

$I(p) = \mathbf{t}, I(q) = \mathbf{f}, I(r) = \mathbf{t}, I(s) = \mathbf{f}$. I is not a $\mathcal{B}_{\mathrm{st}}$-model of

$$\begin{array}{ll} p \leftarrow {\sim}q & {\sim}p \leftarrow q, r \\ q \leftarrow {\sim}p & {\sim}p \leftarrow q, {\sim}s \\ p \leftarrow s, {\sim}r & {\sim}q \leftarrow p \\ r \leftarrow r & {\sim}r \leftarrow {\sim}r \end{array}$$

r only has one justification.

$$\mathbf{t} = I(r) \neq \mathrm{SV}_{\mathcal{B}_{\mathrm{st}}}(r, I) = \mathbf{f}$$

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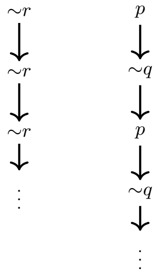

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DEFINITIONS: SUMMARY

- A justification frame contains a set of rules
- The rules determine which justifications are possible
- A branch evaluation determines which branches are “good”
- A justification is “good” if all its branches are “good”
- An interpretation is a model (according to some semantics, induced by the branch evaluation) if the truth value of each fact equals the value of its best justification

OUTLINE

TWO DEFINITIONS OF JUSTIFICATIONS

Definition ([Den93, MBDH22])

Let $\mathbb{JF} = \langle \mathbb{F}, \mathbb{F}_d, R \rangle$ be a justification frame. A (tree-like) justification J in $\mathbb{JF}$ is a labeled tree such that the set of children of each internal node is a case (rule) in R for that node.

Definition ([Mar09, DBS15, MPBD18, MBD21])

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TWO DEFINITIONS OF JUSTIFICATIONS

Definition ([Den93, MBDH22])

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Theorem

Every graph-like justification J_g can be "unfolded" into a tree-like justification J_t such that $\mathrm{val}_{\mathcal{B}}(J_t, x, I) \geq_t \mathrm{val}_{\mathcal{B}}(J_g, x, I)$ for each x

THE GRAPH-REDUCIBILITY PROBLEM

Open Problem (The Graph-Reducibility Problem)

Under which conditions on $\mathcal{B}$ can every tree-like justification J_t be "reduced" to a graph-like justification J_g with $\mathrm{val}_{\mathcal{B}}(J_g, x, I) \geq_t \mathrm{val}_{\mathcal{B}}(J_t, x, I)$?

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Example

The branch evaluation $\mathcal{B}_{\mathrm{ex}}$ maps:

- finite branches to their last element
- infinite branches with a consistent positive tail to $\mathbf{f}$
- infinite branches with a consistent negative tail to $\mathbf{t}$
- other branches to $\mathbf{u}$

consistent branch: whenever $x_i = x_j$ also $x_{i+1} = x_{j+1}$

THE GRAPH-REDUCIBILITY PROBLEM: EXAMPLE

$a \leftarrow b$	
$a \leftarrow c$	$\sim a \leftarrow \sim b, \sim c$
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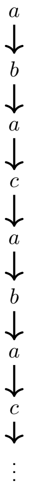

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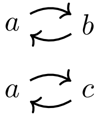

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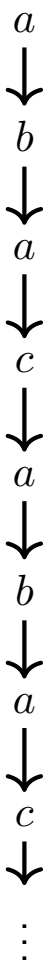

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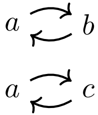

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TREE-LIKE AND GRAPH-LIKE JUSTIFICATIONS

- Our results are only about tree-like justifications

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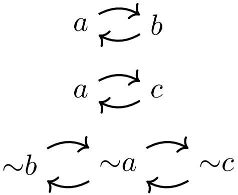
- Our results are only about tree-like justifications
- In examples, I might sometimes draw graph-like justifications, but mean their tree-unfolding

OUTLINE

1. Justification Theory: Motivation & Definitions
2. Two Flavours of Justifications
3. The Consistency Problem
4. The Proof
5. Conclusion

THE CONSISTENCY PROBLEM: EXAMPLE

Graph-like:



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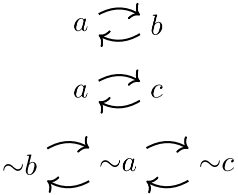

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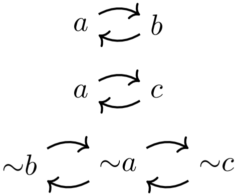

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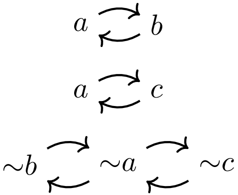

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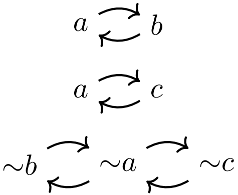

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Can it be so that $\mathrm{SV}(x) >_t \sim \mathrm{SV}(\sim x)$?

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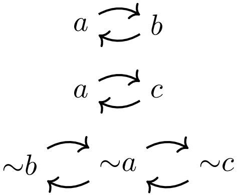

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Can it be so that $\text{SV}(x) >_t \sim\text{SV}(\sim x)$? This would be problematic, it would mean we can explain both a and $\sim a$.

Theorem (Easy — tree-like and graph-like)

Under reasonable assumptions, $\text{SV}(x) \leq_t \sim\text{SV}(\sim x)$ *for all* x.

REASONABLE ASSUMPTIONS

- Set of rules is complementary

- Branch evaluation respects negation

Without these conditions, we are in the wild.

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Definition (Complementary — various other characterizations exist)

We call $\mathbb{JF}$ complementary if for every $x \in \mathbb{F}_d$:

1. for every selection function s for x, there exists an $A \in \mathbb{JF}(\sim x)$ with $A \subseteq \sim \mathrm{Im}(s)$;
2. for every $A \in \mathbb{JF}(x)$, there exists a selection function s for $\sim x$ in R with $\sim \mathrm{Im}(s) \subseteq A$.

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Definition

A branch evaluation $\mathcal{B}$ respects negation if $\mathcal{B}(\sim\mathbf{b}) = \sim\mathcal{B}(\mathbf{b})$ for each branch $\mathbf{b}$.

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Theorem (Our Result)

Every reasonable$^{(*)}$ justification system is tree-like consistent.
$(*)$ *Rules are complementary, and branch evaluaiton respects negation.*

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- Graph-like consistency implies graph-reducibility (and tree-like consistency) [MBD20]

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- Since $\mathcal{B}$ respects negation the value of each branch in J' is at least $\sim v$.

PROOF: BASIS

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A branch selection for x in $\mathbb{JF}$ is a set $\mathbb{B}$ of branches starting in x such that for each locally complete justification J rooted in x, $\mathbb{B}$ contains at least one J-branch.

Lemma

Let $\mathbb{B}$ be a branch selection for x, and let $x \leftarrow A$ be a rule in $\mathbb{JF}$. There exists a $y^* \in A$ such that for every justification $J_{y^*} \in \mathfrak{J}(y^*)$ rooted in y^*, $\mathbb{B}$ contains at least one branch of the form $x \to y^* \to \mathbf{b}$ with $y^* \to \mathbf{b}$ a branch starting from the root of J_{y^*}.

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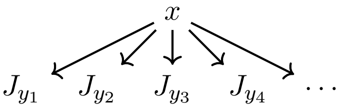

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Corollary

Let $\mathbb{B}$ be a branch selection for x, and let $x \leftarrow A$ be a rule in $\mathbb{JF}$. There exists a $y^* \in A$ such that $\mathbb{B}_{y^*} := \{y^* \to \mathbf{b} \mid x \to y^* \to \mathbf{b} \in \mathbb{B}\}$ is a branch selection for y^*.

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Let $\mathbb{B}$ *be a branch selection for* x*, and let* $x \leftarrow A$ *be a rule in* $\mathbb{JF}$*. There exists a* $y^* \in A$ *such that* $\mathbb{B}_{y^*} := \{y^* \to \mathbf{b} \mid x \to y^* \to \mathbf{b} \in \mathbb{B}\}$ *is a branch selection for* y^**.*

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Corollary

If $\mathbb{JS}$ is complementary and $\mathbb{B}$ is a branch selection for x, then there is a rule $\sim x \leftarrow B_{\sim x}$ such that for each $y \in \sim B_{\sim x}$, $\mathbb{B}_y := \{y \to \mathbf{b} \mid x \to y \to \mathbf{b} \in \mathbb{B}\}$ is a branch selection for y.

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Idea of the proof of the main theorem:

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Idea of the proof of the main theorem:

- Given a branch selection for x, construct a justification for $\sim x$. How?

THE PROOF

Corollary

Let $\mathbb{B}$ be a branch selection for x, and let $x \leftarrow A$ be a rule in $\mathbb{JF}$. There exists a $y^* \in A$ such that $\mathbb{B}_{y^*} := \{y^* \to \mathbf{b} \mid x \to y^* \to \mathbf{b} \in \mathbb{B}\}$ is a branch selection for y^*.

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If $\mathbb{JS}$ is complementary and $\mathbb{B}$ is a branch selection for x, then there is a rule $\sim x \leftarrow B_{\sim x}$ such that for each $y \in \sim B_{\sim x}$, $\mathbb{B}_y := \{y \to \mathbf{b} \mid x \to y \to \mathbf{b} \in \mathbb{B}\}$ is a branch selection for y.

Idea of the proof of the main theorem:

- Given a branch selection for x, construct a justification for $\sim x$. How?
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Actual proof.... a bit finicky, but this is the gist

OUTLINE

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Interested? I'm looking for good postdocs/PhD students

Thanks for your attention!

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